JOURNAL OF APPROXIMATION THEORY **58**, 55–57 (1989)

A Remark on the Sum of Proximinal Subspaces

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Communicated by Frank Deutsch

Received April 1, 1987

A subspace M of a Banach Space X is said to be *proximinal* if for every x in X, $\inf\{\|x-m\|: m \in M\} = d(x, M)$ is attained. E. W. Cheney and D. E. Wulbert [C-W] asked the following question:

Question 0. If U and V are proximinal subspaces of a Banach space X and if $U+V$ is closed, does it follow that $U+V$ is proximinal?

M. Feder [F] showed the answer to the above question is negative. Indeed, he proved the following theorems.

THEOREM A. *There is a Banach space which contains nonreflexive proximinal subspaces U and V such that $U+V$ is closed and $U+V$ is not proximinal.*

THEOREM B. *Let U and V be subspaces of a Banach space X, Assume that U is proximinal, V is reflexive, $U \cap V$ is finite dimensional, and that $U+V$ is closed, Then $U+V$ is proximinal.*

One may ask the following questions:

Question 1. Let U be a nonreflexive Banach space. Is there a Banach space X which contains U as a subspace and satisfies the following conditions:

(i) U is a proximinal subspace of X,

(ii) X contains a proximinal subspace V such that $U+V$ is closed, but $U+V$ is not proximinal?

Question 2. In Theorem B, can the assumption, "$U \cap V$ is finite dimensional," be dropped?

In this article, we show that the answers of both questions above are positive.

First, we need the following well-known result.

0021-9045/89 $3.00

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LEMMA 1 (Theorem 2.15 in [S]). *A normed linear space G is proximinal in every superspace E if and only if G is a reflexive Banach space, Moreover, if G is a nonreflexive Banach space, then G can be embedded isometrically as a nonproximinal closed hyperplane in another Banach space E.*

Suppose U is a nonreflexive Banach space. By Lemma 1, there is a Banach space $E = \operatorname{span}\{U, e\}$ such that U is a nonproximal closed hyperplane in E (i.e., $\inf\{\|e-u\|: u \in U\}$ is not attained). Let $(E \oplus E)_\infty$ be the Banach space

$$\{(x, y\}: x \in E, y \in E\}$$

with the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$. Let $Y = U \oplus 0$, $Z = 0 \oplus U$, and $X = \operatorname{span}\,(Y \cup Z \cup \{e \oplus e\})$. We claim that Y and Z are proximinal subspaces of X, and that $Y + Z$ is closed but not proximinal in X.

(a) Y is proximinal in X. Given $y' \in Y$, $z \in Z$, and λ a scalar, we must show that there is $y \in Y$ such that $d(y' + z + \lambda(e \oplus e), Y) = \|y' + z + \lambda(e \oplus e) - y\|$. Since $y' \in Y$, we may assume $y' = 0$. If $\lambda = 0$, then we choose $y = 0$ ($d(z, Y) = \|z\|$ for any $z \in Z$). By homogeneity, we may assume $\lambda = 1$. Since U is not proximinal, for any $u \in U$

$$\|e + u\| > d(e, U).$$

So if $z = 0 \oplus z$, then $\|e + z\| > d(e, U)$. Let u be a vector in U such that $\|e - u\| \leqslant \|e + z\|$ and let $y = u \oplus 0$. Then $d(z + (e \oplus e), Y) = \|z + (e \oplus e) - y\|$. So Y is proximinal. Similarly, Z is proximinal.

(b) $Y + Z$ is not proximinal. Clearly,

$$d(e \oplus e, Y + Z) = d(e, U).$$

But U is nonproximinal, so $\inf\{\|e \oplus e - x\|: x \in Y + Z\}$ is not attained.

Remark 1. It is known that if U is a noncomplete norm space, then U can be embedded isometrically as a nonproximinal closed hyperplane in another norm space. So the above result is still true if U is noncomplete.

To answer the second question, we need the following lemma which was proved by E. W. Cheney and D. E. Wulbert.

LEMMA 2 (Proposition 5(3) in [C-W]). *Let X be a normed linear space and let Y and Z be subspaces of E such that $Z \subseteq Y$. If Z is proximinal in X and Y/Z is proximinal in X/Z, then Y is proximinal.*

So the question which E. W. Cheney asked is equivalent to:

Question 0′. Suppose U and V are proximinal in X. Is $U+V/V$ proximinal in X/V?

By Lemma 1, we may ask under what conditions $U+V/V$ is reflexive. It is well known that $U/U\cap V$ is algebraically isomorphic to $U+V/V$ and the mapping T: $U/U\cap V\to U+V/V$

$$T(u+(U\cap V))=u+V$$

has norm less than one. If $U+V$ and X are Banach spaces by the open mapping theorem, $U/U\cap V$ and $U+V/V$ are isomorphic. So if U is reflexive, then $U+V/V$ is reflexive. We have the following theorem.

THEOREM 3. *Let U be a Banach space. Then U is reflexive if and only if for any Banach space $X\supseteq U$, V is proximinal in X and $U+V$ is closed implies $U+V$ is proximinal in X.*

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